Akademie věd České republiky

Teze disertace k získání vědeckého titulu „doktor věd“
ve skupině věd fyzikálně-matematických

Graph limits

Komise pro obhajoby doktorských disertací v oboru
Matematické struktury

Jméno uchazeče: Jan Hladký

Pracoviště uchazeče: Ústav informatiky AV ČR, v. v. i.

Místo a datum: Praha, 5. dubna 2024

Synopsis

The thesis contains 17 papers of the author written jointly with 22 coauthors revolving around the theory of dense graph limits (a.k.a. graphons).

Three papers in Part I study the relation between the cut distance which underlies the topology of graphons and the weak* topology. This leads in particular to alternative proofs of the Lovász–Szegedy compactness theorem. The fourth paper deals with norms defined by homomorphism densities, an recently emerged area which connects functional analysis and extremal graph theory. It answers a question of Hatami about moduli of convexity and smoothness of weakly norming graphs. The second result in the paper is a strong factorization result for norms defined by disconnected graphs. This factorization results in fact corrects a number of imprecisions which existed in literature in the theory of graph norms regarding disconnected graphs.

Part II introduces and studies counterparts to concepts well known in finite graphs, such as independent set, clique, coloring, matching, and tiling in the setting of graphons. In particular, for the concepts of matchings and tilings, we are able to translate much of the powerful theory of linear programming to the graphon setting. As an application, we reprove the tiling theorem of Komlós, adding a stability part to it. The last paper in Part II deals with fractional isomorphism. Fractional isomorphism is a concept that existed for finite graphs for four decades and its graphon version was recently worked out by Grebík and Rocha. We prove that fractionally isomorphic graphons can be approximated by fractionally isomorphic finite graphs, thus linking tightly the connection between the graph and the graphon concept and answering the main question of Grebík and Rocha.

Part III deals random structures connected with graphons. Two papers give results on $\mathbb{G}(n, W)$ which is a graphon generalization of the well studies Erdős–Rényi random graph model $\mathbb{G}(n, p)$. In particular, we obtain a counterpart for $\mathbb{G}(n, W)$ of the classical results of Matula, and independently Grimmett and McDiarmid from the 1970s about the clique number in $\mathbb{G}(n, p)$. Another paper in Part III deals with the uniform spanning tree of a graph. This is a basic model in probability theory with important links to statistical physics. In particular, four decades ago, Kolchin and independently Grimmett proved that locally the uniform spanning tree of the complete graph K_n converges to the Galton–Watson branching tree with offspring distribution $Poisson(1)$ conditioned on survival, as $n \to \infty$. We provide a counterpart to this statement for uniform spanning trees on sequences of finite graphs $(G_n)_n$, which converge to a graphon W. In particular, we show that the local limit distribution of the uniform spanning tree corresponds to a certain inhomogeneous branching process derived from W. The last two papers in Part III introduce a new theory, which we call "flip processes". Flip processes are a general framework of graph dynamics, which in particular extends the Erdős–Rényi graph and the triangle-removal graph process. Each flip process is given by a "rule" which is a set of transformations on k-vertex graphs. Starting with an initial graph, in each step of the dynamics, a random k-tuple is sample from the graph and is replaced according to the rule. The main result is that typical behavior of such flip processes on finite graphs can be modeled by real-time trajectories on the space of graphons.

Part IV deals with other theories of limits of discrete structures. Two papers are contributions to existing limit theories, of partially ordered sets, and permutations, respectively. The theory of limits of partially ordered sets was introduced by Janson who gave a certain construction of limit objects on some *ad hoc* measure space equipped with a partial order. We give a more canonical construction by constructing the limit on $[0, 1]$ with the standard order. Our contribution to the theory of permutation limits is in the area of pattern-avoidance. This is arguably the most important direction of research in combinatorial study of permutations. Our result about permutons — limit counterpart to permutations — says that any pattern-avoiding permuton has necessarily one-dimensional structure, compared to generally two-dimensional structure of permutons. The third paper introduces a new limit theory, namely that of Latin squares.

Introduction

This document is a summary to my thesis *Graph limits* to submitted for the DSc degree at the Czech Academy of Sciences. The thesis itself consists of the following 17 papers.

- In Part I: *Topologies on the graphon space*,

  [P1] Martin Doležal, Jan Hladký: *Cut-norm and entropy minimization over weak* limits*, Journal of Combinatorial Theory, Series B 137 (2019), 232–263.

  [P2] Martin Doležal, Jan Grebík, Jan Hladký, Israel Rocha, Václav Rozhoň: *Relating the cut distance and the weak* topology for graphons.* Journal of Combinatorial Theory, Series B 147 (2021), 252–298.

  [P3] Martin Doležal, Jan Grebík, Jan Hladký, Israel Rocha, Václav Rozhoň: *Cut distance identifying graphon parameters over weak* limits.* Journal of Combinatorial Theory, Series A 189 (2022), Paper No. 105615, 57 pp.

  [P4] Frederik Garbe, Jan Hladký, Joonkyung Lee: *Two remarks on graph norms.* Discrete and Computational Geometry 67 (2022), no. 3, 919–929.

- In Part II: *Classical graph-theoretic concepts in graphons*,

  [P5] Jan Hladký, Ping Hu, Diana Piguet: *Tilings in graphons.* European Journal of Combinatorics 93 (2021), Paper No. 103284, 23 pp.

  [P6] Jan Hladký, Ping Hu, Diana Piguet: *Komlós's tiling theorem via graphon covers.* Journal of Graph Theory 90 (2019), no. 1, 24–45.

  [P7] Martin Doležal, Jan Hladký: *Matching polytons.* Electronic Journal of Combinatorics 26 (2019), no. 4, Paper No. 4.38, 33 pp.

  [P8] Jan Hladký, Israel Rocha: *Independent sets, cliques, and colorings in graphons.* European Journal of Combinatorics 88 (2020), 103108, 18 pp.

  [P9] Jan Hladký, Eng Keat Hng: *Approximating fractionally isomorphic graphons.* European Journal of Combinatorics 113 (2023), 103751, 19 pp.

- In Part III: *Inhomogeneous random structures from graphons*,

[P10] Jan Hladký, Christos Pelekis, Matas Šileikis: *A limit theorem for small cliques in inhomogeneous random graphs.* Journal of Graph Theory 97 (2021), no. 4, 578–599.

[P11] Martin Doležal, Jan Hladký, András Máthé: *Cliques in dense inhomogeneous random graphs.* Random Structures and Algorithms 51 (2017), no. 2, 275–314.

[P12] Jan Hladký, Asaf Nachmias, Tuan Tran: *The local limit of the uniform spanning tree on dense graphs.* Journal of Statistical Physics 173 (2018), no. 3-4, 502–545.

[P13] Frederik Garbe, Jan Hladký, Matas Šileikis, Fiona Skerman: *From flip processes to dynamical systems on graphons*, Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques, accepted, available on journal webpage.

[P14] Pedro Araújo, Eng Keat Hng, Jan Hladký, Matas Šileikis: *Prominent examples of flip processes*, Random Structures and Algorithms, 64 (2024), no. 3, 692–740.

- In Part IV: *Limits of other discrete structures*,

[P15] Frederik Garbe, Robert Hancock, Jan Hladký, Maryam Sharifzadeh: *Limits of Latin squares*, Discrete Analysis 2023:8, 66pp.

[P16] Jan Hladký, András Máthé, Viresh Patel, Oleg Pikhurko: *Poset limits can be totally ordered*, Transactions of the American Mathematical Society, Volume 367 (2015), pages 4319-4337.

[P17] Frederik Garbe, Jan Hladký, Gábor Kun, Kristýna Pekárková: *On pattern-avoiding permutons*, Random Structures and Algorithms, DOI: 10.1002/rsa.21208.

These results concern the theory of dense graph limits. The idea of this theory, initiated by Lovász and Szegedy [25] and by Borgs, Chayes, Lovász, Sós and Vesztergombi [6], is in compactifying the space of isomorphism types of finite graphs.[1] More precisely, Lovász and

[1] All finite graphs are considered simple (that is, each pair of vertices induces 0 or 1 edges), undirected and without self-loops. While the assumption about self-loops is insignificant and a limit theory of directed graphs can be created *mutatis mutandis*, the simplicity assumption is important and limit theories of sequences multigraphs (of unbounded edges multiplicities in particular) are more subtle.

Szegedy introduce the notion of *graphons*, which are defined as symmetric Lebesgue measurable functions $W : [0,1]^2 \to [0,1]$. We write $\mathcal{W}_0$ for the set of graphons and λ for the Lebesgue measure. Each finite graph can be represented by a graphon by simply taking its adjacency matrix and squashing it into the unit square. In particular, a graphon representation is $\{0,1\}$-valued and constant on squares the whose side-length is the inverse of the number of vertices. The construction of a compact metric is done in two steps. First, we define the *cut norm* of a symmetric function $Z \in L^\infty\left([0,1]^2\right)$ as

$$\|Z\|_\square = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} Z \right| ,$$

where the supremum ranges over all measurable sets $S,T \subset [0,1]$. The assumption of measurability of sets and functions will be implicit below. The compactness theorem then reads as follows. Let $d_\square(\cdot,\cdot)$ be the induced distance, the so-called *cut norm distance*. Then we define the *cut distance* between two graphons U and W by

$$\delta_\square(U,W) = \inf_\pi \|U - W^\pi\|_\square ,$$

where the infimum ranges over all measure preserving bijections $\pi : [0,1] \to [0,1]$ and we define $W^\pi(x,y) := W(\pi(x),\pi(y))$. A particular feature is that if we have graphons U and W obtained by squashing adjacency matrices of the same graph which just differ by the order in which the vertices were enumerated, we have $\delta_\square(U,W) = 0$. This feature does away with the potential issue of nonuniqueness of the adjacency matrix. The compactness theorem of Lovász and Szegedy then reads as follows.

Theorem 1. *For each sequence of graphs $G_1, G_2, \ldots$ there exists a subsequence $G_{n_1}, G_{n_2}, \ldots$ and a graphon W so that $\delta_\square(G_{n_i}, W) \to 0$ as $i \to \infty$.*

Much of the strength of the theory of graph limits stems from this single result. That is, the compactness theorem allows to transfer statements from the category of finite graphs (in particular, when they are of asymptotic nature) to the category of graphons. The advantages of working with the latter is in availability of various analytic tools. This idea follows a common framework in mathematics, which underlies even the most basic concepts. We illustrate this on one of the most fundamental concepts in mathematics — that of real numbers. Indeed, much of the motivation of introducing the real numbers is that computations — even those that are phrased purely in terms of rational numbers — become more tractable in this bigger completion due to availability of tools such as differential and integral calculus. Returning to the theory of graph limits and the compactness theorem, note that it is crucial that many important graph parameters (and their extensions to graphons) are continuous with respect to the cut distance.

Part I: *Topologies on the graphon space*

Papers [P1], [P2], and [P3] create a framework that connects the cut distance and the weak* topology. I will describe these three papers, and return to [P4] later. By routinely unfolding the definition of the cut distance, we see that convergence of graphons $W_1, W_2, \ldots$ to a graphon W in it amounts to the existence of a sequence of measure preserving bijections $\pi_1, \pi_2, \ldots : [0,1] \to [0,1]$ such that

$$\limsup_n \sup_{S,T\subset[0,1]} \left| \int_{S\times T} W_n^{\pi_n} - W \right| = 0 \ . \tag{1}$$

This formula is similar to convergence in weak* topology when considering the predual of L^1-functions on $[0,1]^2$. Indeed, such a convergence amounts to

$$\sup_{S,T\subset[0,1]} \limsup_n \left| \int_{S\times T} W_n^{\pi_n} - W \right| = 0 \ . \tag{2}$$

In particular, it is an easy exercise that the former implies the latter. The weak* topology (on uniformly bounded functions, which graphons are) is compact by the Banach–Alaoglu theorem. So, if it were, that the two metrics were equivalent, we would get Theorem 1 for free as a consequence. Alas, taking $(W_n)_n$ to be the system of 2-dimensional Rademacher functions (a.k.a. chessboards) and $W \equiv \frac{1}{2}$, we get an example of functions satisfying (2) (even with $\pi_1 = \pi_2 = \ldots = id$) but not (1).

The idea behind [P1] is that while the weak* convergence condition (2) itself is not sufficient to identify cut distance limits, it may be useful to narrow the search space for them. More specifically, we introduce the entropy function $Ent : \mathcal{W}_0 \to [0,1]$ by

$$Ent(W) := \int_x \int_y H(W(x,y)) \ ,$$

where H is the binary entropy function,

$$H(t) = -t \log_2 t - (1-t) \log_2 (1-t) .$$

Given a sequence of graphons $W_1, W_2, \ldots$, we define $\mathrm{ACC}\,(W_1, W_2, \ldots)$ as the set of accumulation points in the weak* topology over all sequences $W_1^{\pi_1}, W_2^{\pi_2}, \ldots$. The main result of [P1] then reads as follows.

Theorem 2. *Suppose that $W_1, W_2, \ldots$ is a sequence of graphons and $W \in \mathrm{ACC}\,(W_1, W_2, \ldots)$ is such that $Ent(W) = \inf_{U \in \mathrm{ACC}(W_1, W_2, \ldots)} Ent(U)$. Then W is a cut distance accumulation point of $W_1, W_2, \ldots$.*

Hence Theorem 2 almost reproves Theorem 1. The only issue is that we are not guaranteed the existence of the entropy minimizer. That is, while the Banach–Alaoglu theorem guarantees that $\mathrm{ACC}\,(W_1, W_2, \ldots) \neq \emptyset$, we do not have a guarantee that the infimum of entropies is attained. We surmount this shortcoming by the following result, also given in [P1].

Proposition 3. *Suppose that $W_1, W_2, \ldots$ is a sequence of graphons. Then there exists a subsequence $W_{n_1}, W_{n_2}, \ldots$ such that there exists a graphon $W \in \mathrm{ACC}\,(W_{n_1}, W_{n_2}, \ldots)$ with the property that*

$$Ent(W) = \inf_{U \in \mathrm{ACC}\left(W_{n_1}, W_{n_2}, \ldots\right)} Ent(U) .$$

Obviously, applying first Proposition 3 and then Theorem 2 implies Theorem 1. (Let us remark, that our proof was not the only alternative to Theorem 1, another proof using nonstandard analysis was given by Elek and Szegedy [10], whose main aim was to develop limit theory of hypergraphs.) In time of writing [P3], we viewed Proposition 3 merely as a technical nuisance. I will return to this later.

Papers [P2] and [P3] were written at around the same time, extending [P1] in two different ways. Paper [P2] takes an abstract perspective. With that perspective, every graphon is rather seen through its *envelope*, $\langle W \rangle = \mathrm{ACC}\,(W, W, \ldots)$. Recall that given a metric space (X, d), there is a notion of *Vietoris hyperspace* over X, which is a space whose points are nonempty closed sets of X, and distance between two of them is defined as

$$d_{VH}(A, B) = \max \left(\max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right) .$$

It is a well-known fact that the Vietoris hyperspace of a compact Hausdorff space is again compact. We take this general construction to $X = \mathcal{W}_0$ equipped with (a metrization of) the weak* topology. The closed sets we pay particular attention to are the envelopes. The importance of this point of view is shown by the following theorem, proven in [P2].

Theorem 4. *Suppose that W is a graphon and that $W_1, W_2, \ldots$ is a sequence of graphons. Then the sequence of envelopes $\langle W_1 \rangle, \langle W_2 \rangle, \ldots$ converges to the envelope $\langle W \rangle$ in the Vietoris hyperspace over the weak* topology if and only if $W_1, W_2, \ldots$ converges to W in the cut distance.*

Let us now give another main result of [P2]. This time we do not directly work with the concept of envelopes. Rather, we work with a set $\mathrm{LIM}(W_1, W_2, \ldots)$ which is defined the same as $\mathrm{ACC}(W_1, W_2, \ldots)$ but using limit points rather than accumulation points.

Theorem 5. *Suppose that $W_1, W_2, \ldots$ is a sequence of graphons. Then there is a subsequence $W_{n_1}, W_{n_2}, \ldots$ such*

$$\mathrm{ACC}(W_{n_1}, W_{n_2}, \ldots) = \mathrm{LIM}(W_{n_1}, W_{n_2}, \ldots) \ .$$

Theorem 5 generalizes Proposition 3 since it can be easily shown that $\mathrm{LIM}(W_{n_1}, W_{n_2}, \ldots)$ is closed in L^1 and thus an entropy minimizer exists. But so far, it could be regarded again just as a convenient auxiliary result. The next theorem from [P2] however shows the combinatorial relevance of sequences with ACC = LIM.

Theorem 6. *A sequence of graphons $W_1, W_2, \ldots$ is Cauchy in the cut distance if and only if $\mathrm{ACC}(W_1, W_2, \ldots) = \mathrm{LIM}(W_1, W_2, \ldots)$. Furthermore, in this case the limit graphon W satisfies $\mathrm{LIM}(W_1, W_2, \ldots) = \langle W \rangle$.*

We remark that some of the result from [P2] could be derived from the theory of "multiway cuts" worked out in [7]. However, we believe that our view is a more conceptual one, and directly suited to the analytic setting.

Let us now turn to [P3]. The key concept is that of *cut distance identifying graphon parameters.* More precisely, we say that a graphon parameter $f : \mathcal{W}_0 \to \mathbb{R}$ is cut distance identifying if for every sequence of graphons $W_1, W_2, \ldots$ and for every $W \in \mathrm{LIM}(W_1, W_2, \ldots)$ we have that $f(W)$ is the minimum (or maximum, depending on the context) of $f(U)$ over $U \in \mathrm{LIM}(W_1, W_2, \ldots)$. In that language, the main result of [P1] is that the parameter $Ent(\cdot)$ is cut distance identifying. In [P3] we establish that several other important graph parameters are cut distance identifying. Here, we recall the two most important such results, one concerning norming graphs and the other concerning spectra.

To explain the first result, we need to recall the recall the notion of graph norms. These were introduced by Hatami in [16]. Given a graph H and a symmetric bounded function $U : [0,1]^2 \to \mathbb{R}$, we define $\|U\|_H := \sqrt[e(H)]{t(H,U)}$ and $\|U\|_{r(H)} := \sqrt[e(H)]{|t(H,U)|}$, where $t(H,U)$ is the usual homomorphism density,

$$t(H,U) = \int_{\mathbf{x}\in[0,1]^{V(H)}} \prod_{ij\in E(H)} U(x_i,x_j) \ .$$

We say that H is *norming* if $\|\cdot\|_H$ is a norm, and that H is *weakly norming* if $\|\cdot\|_{r(H)}$ is a seminorm. We prove that for each norming graph H, the parameter $t(H,\cdot)$ is cut distance identifying. By further arguments, we can actually resolve a conjecture of Kráľ, Martins, Pach, and Wrochna [23] which concerns two extremal concepts stemming from "Sidorenko's conjecture". Recall that this conjecture from the 1993 (made in a slightly different form also by Erdős and Simonovits in 1983) asserts that for each bipartite graph H and for each graphon W we have $t(H,W) \geq t\left(H, W^{\Join\{[0,1]\}}\right)$. Here, for $\mathcal{P}$ a finite partition of $[0,1]$ into sets of positive measure, we define the *stepping* $W^{\Join\mathcal{P}} = \mathbb{E}\left[W \,\vdots\, \mathcal{P}\times\mathcal{P}\right]$. This conjecture was strengthened into the "Forcing conjecture" of Skokan and Thoma, [31]: For each bipartite graph H which is not a forest we have $t(H,W) > t\left(H, W^{\Join\{[0,1]\}}\right)$ unless W is a constant graphon. It is easy that in both cases the opposite directions "Sidorenko⇒bipartite" and "forcing⇒bipartite non-forest" are trivial. Both conjectures are among the most important open problems in extremal graph theory. The reason Hatami introduced graph norms was his observation that every weakly norming graph is Sidorenko, and further, if $\|\cdot\|_H$ is uniformly convex – a property he established in the same paper for all norming graphs – then the the graph is forcing. In fact, it turns out it is more useful to introduce concepts of "step-Sidorenko" and "step-forcing" properties, as was done in [23]. We say that a graph H is *step-Sidorenko* if $t(H,W) \geq t\left(H, W^{\Join\mathcal{P}}\right)$ for every $\mathcal{P}$ a finite partition of $[0,1]$ and every graphon W. We say that H is *step-forcing* if $t(H,W) > t\left(H, W^{\Join\mathcal{P}}\right)$ for every $\mathcal{P}$ a finite partition of $[0,1]$ and every graphon W with $W \neq W^{\Join\mathcal{P}}$. Our main connection between graph norms and step-Sidorenko/-forcing properties from [P3] goes as follows.

Theorem 7.

- *Suppose that H is a connected graph. Then H is step-Sidorenko if and only if it is weakly norming.*
- *Suppose that H is a graph. If H is norming then it is step-forcing.*

Let us now turn to the spectral result. We recall the theory of graphon spectra briefly. Each graphon W can be viewed as an operator from $L^2([0,1])$ to itself defined by $Wf(x) = \int_y W(x,y)f(y)$. This is a Hilbert–Schmidt operator and thus has a real spectrum which is an at most countable set in $[-1,1]$ with only possibly accumulation point at 0. Our result is that the spectrum is a cut distance identifying. Obviously, the precise definition has to exceed that of (one-dimensional) "parameters". So, what we prove is that if W is a cut distance limit of $W_1, W_2, \ldots$ and $U \in \mathrm{LIM}(W_1, W_2, \ldots)$ then for the eigenvalues of W,

$$\lambda_1^+ \geq \lambda_2^+ \geq \ldots > 0 > \ldots \geq \lambda_2^- \geq \lambda_1^-$$

and for the eigenvalues of U,

$$\kappa_1^+ \geq \kappa_2^+ \geq \ldots > 0 > \ldots \geq \kappa_2^- \geq \kappa_1^-$$

we have $\lambda_i^+ \geq \kappa_i^+$ and $\lambda_i^- \leq \kappa_i^-$ for every i. Further, at least one of these inequalities is strict of U is not a cut distance limit of $W_1, W_2, \ldots$.

The last important topic from [P3] is a connection between the theory cut distance identifying graphon parameters and regularity lemmas (and the Frieze–Kannna regularity lemma from [12] in particular).[2] Recall that in the heart of proofs of regularity lemmas is the "index-pumping lemma". The "index" here is traditionally taken to be the L^2-norm and the key argument is that when refining an irregular partition, the index goes up. We show that any cut distance identifying graph parameter can be taken as the index. This includes homomorphism densities of each norming graph, such as the 4-cycle.

Let us now turn to the last a short paper [P4]. It contains two results concerning norming and weakly norming graphs. The first result is about moduli of convexity and smoothness. These are classical notions in functional analysis that describe geometric properties of a Banach space. The Banach space in question is the linear extension $\mathcal{W}$ of the space of graphons. Equivalently, $\mathcal{W}$ consists of all symmetric functions in $L^\infty\left([0,1]^2\right)$. The first result in this direction, obtained in the seminal

[2] First comments on such a connection appear already in [P1].

paper [16], says that if H is a norming graph with m edges, then the modulus of convexity and the modulus of smoothness of $\|\cdot\|_{r(H)}$ are (up to a constant factor) the same as for the L^m-norm. The L^m-norm is well-understood, and in particular is uniformly convex and uniformly smooth. Answering an open problem from [16], we prove the following negative result.

Theorem 8. *Suppose that H is a weakly norming graph. Then $\|\cdot\|_{r(H)}$ is not uniformly convex nor uniformly smooth in $\mathcal{W}$.*

We now turn to the second result of [P4]. The result fits in a broad framework of trying to understand how one can compose a norming graph from several smaller norming graphs, or dually, how one can decompose a norming graph into several smaller norming graphs. The simplest such composition concept is that of taking a disjoint union. It is trivial to check that if H is a norming graph and G consists of any number of disjoint copies of H, then G is again norming, and $\|\cdot\|_H = \|\cdot\|_G$. The same can be seen for weakly norming graphs. We prove that this is the only of creating norming or weakly norming graphs with several connected components. Obviously, from this characterization, we need to discard isolated vertices, which play no role in the definition of homomorphism density and hence of graph norms.

Theorem 9. *Suppose that F is a disconnected graph without isolated vertices.*

- *If F is norming then all the components of F are isomorphic to the same norming graph.*
- *If F is weakly norming then all the components of F are isomorphic to the same weakly norming graph.*

This result corrects a number of omissions in standard texts including [24]. These texts worked with assumptions such as "Take an arbitrary norming graph F. Without loss of generality, suppose that F is connected." It is only Theorem 9 that indeed justifies that these assumptions can be made.

Part II: *Classical graph-theoretic concepts in graphons*

This part explores the connection between concepts and parameters that are extensively studied in graphs, and their graphon counterpart.

Since the theory of graphons is fairly young, some even basic notions were not developed for graphons prior to our work.

Papers [P7] and [P5] introduce the concept of the matching ratio and F-tiling ratio for graphons. The notion of matchings is standard and needs to introduction but it is worth recalling that of F-tilings. Given graphs F and G, an *F-tiling* in G is a collection of vertex-disjoint copies of F. These copies are not required to be induced. The *F-tiling number of* G, denoted $til(F, G)$, is the maximum size of an F-tiling. When $F = K_2$, we get the concepts of matchings and of the matching number.

In the setting of dense graphs, we will be looking at F fixed, G large, and introduce the rescaling $\frac{til(F,G)}{v(G)}$. When extending a graph parameter to graphons, we usually want to do it in a fashion which is continuous with respect to the cut distance. This is not possible for the parameter $\frac{til(F,\cdot)}{v(\cdot)}$. Indeed, for $n \in \mathbb{N}$ take G_n to be n disjoint copies of F. It is obvious that $\frac{til(F,G_n)}{v(G_n)} = \frac{1}{v(F)}$ which is as big as $\frac{til(F,\cdot)}{v(\cdot)}$ can get. On the other hand, the sequence $(G_n)_n$ has the constant-0 graphon O as the cut distance limit. Even before having defined the F-tiling ratio of a graphon, it is obvious, that we must have $til(F, O) = 0$, thus showing discontinuity. However, in [P5] we do come up with a reasonable concept of the F-tiling ratio of a graphon, which is lower semicontinuous. I will explain the idea in the simplest case $F = K_2$, where I take the liberty of assuming the reader's familiarity with the notion of fractional matchings, vertex covers, fractional vertex covers, and the relations between them given by the linear programming duality. While the matching number and the fractional matching number can be different in general, the (very broad) message of [17] was that they should be regarded as similar for dense graphs. If that is the case, then, by the the LP duality, this quantity is equal to the minimum fractional vertex cover. And while the concepts of matchings and the fractional matchings are elusive at best for graphons, fractional vertex cover generalizes in a straightforward way. That is, we say that a function $f : [0, 1] \to [0, 1]$ is a *fractional vertex cover* of a graphon W if for almost every $(x, y) \in [0, 1]^2$ we have

$$W(x, y) = 0 \quad \text{or} \quad f(x) + f(y) \geq 1 .$$

To conclude, we define the *matching ratio* of a graphon W as the infimum of $\|f\|_1$ over all fractional vertex covers f of W. In [P5] we show that the matching ratio (and F-tiling ratio) defined in this way enjoys many favorable properties.

In [P7] we work out a graphon counterpart to the matching potytope (and its variants). To give an example of results we prove, recall a well-known theorem in graph theory that says that the vertices of the fractional matching polytope are integral if and only if the graph is bipartite. We prove a graphon counterpart of this result.

Paper [P6] uses the theory of F-tilings to reprove and strengthen a beautiful extremal graph theoretic result of Komlós [22]. Let us give some background. Suppose that F is a fixed graph of chromatic number r, n is large, and G is a graph of order n. It is easy to see (for example from the Erdős–Stone theorem) that if the minimum degree of G is at least $\left(1 - \frac{1}{r-1} + o_n(1)\right) n$ then G contains a copy of F. It is further easy to see that if we change the $o_n(1)$-term to a fixed constant $\epsilon > 0$, then G contains an F-tiling of size linear in n. We can now ask what is the optimal minimum-degree bound for G to contain an F-tiling of a given size. To this end, Komlós came up with a parameter, called the *critical chromatic number* of F. We will not define it here, but the definition is not difficult and the parameter satisfies $\chi_{cr}(F) \in (r-1, r]$. He proved that the chromatic number and the critical chromatic number are the relevant quantities for the minimum-degree for an F-tiling of a given size, where the former plays a more important role for smaller size and the latter takes over for bigger sizes.

Theorem 10. *Suppose that F is fixed, and $\alpha \in (0,1]$ is given. An n-vertex graph G with minimum degree at least*

$$\left((1-\alpha)\cdot\left(1-\frac{1}{\chi(F)-1}\right)+\alpha\cdot\left(1-\frac{1}{\chi_{cr}(F)}\right)\right)n \qquad (3)$$

contains an F-tiling covering $(\alpha - o_n(1))\, n$ vertices of G.

Furthermore, Komlós showed that the bound is best possible. That is, he constructed a graph, called the *bottleneck graph* $B_{n,\alpha,F}$, satisfying the bound (3). The construction is a simple modification of the usual construction of Turán graphs. Komlós' proof was quite unique. While in extremal graph theory, it is very standard to apply the Regularity lemma and work with the cluster graph of the original graph, Komlós applied the Regularity lemma iteratively, thus obtaining "the cluster graph of the cluster graph of ... of G"; the number of iterations depends on the $o_n(1)$-term.

Using the theory of F-tilings in graphons developed in [P5] we reprove Theorem 10 adding a stability counterpart to it. It is worth noting that while sometimes a stability result can be obtained from the original extremal result just by carefully revising the proof, I do not think anything like that would be possible with the iterative regularization of the original proof.

Theorem 11. *Suppose that F is fixed, and $\alpha \in (0,1)$ is given. For each $\epsilon > 0$ there exists an $\delta > 0$ with the following property. An n-vertex graph G with minimum degree at least*

$$\left((1-\alpha)\cdot\left(1-\frac{1}{\chi(F)-1}\right)+\alpha\cdot\left(1-\frac{1}{\chi_{cr}(F)}\right)\right)n$$

contains an F-tiling covering $(\alpha+\delta)n$ vertices of G unless G is ϵ-close to $B_{n,\alpha,F}$ in the edit distance.

We now turn to [P8]. This paper introduces and studies several further classical concepts from graph theory to the setting of graphons. The most important of them is that of an independent set. We say that a set $I \subset [0,1]$ is an *independent set* of a graphon W if W is constant-0 almost everywhere on $I \times I$. The *independence ratio* is then defined as $\alpha(W) = \sup_I \lambda(I)$, where the supremum is over all independent sets I. This definition extends the definition of the independence number of a finite graph and the independence ratio as its natural rescaling. We prove that the independence ratio is lower semicontinuous.

Theorem 12. *The independence ratio is lower semicontinuous in the cut distance topology.*

We now move to the chromatic number. We say that a graphon W is *k-colorable* if there exists a partition $[0,1] = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_k$ into independent sets. Obviously, there is also a nontrivial concept of *countable colorability.* The *chromatic number* $\chi(W)$ is the least k for which k is k-colorable. We prove that the chromatic number is lower semicontinuous. However, the two most important results concerning the chromatic number of graphons in the dissertation come from [P7] and [P11], respectively.

Theorem 13. *A graphon W is k-colorable if and only if for all finite graph H of chromatic number $k+1$ we have $t(H,W) = 0$.*

Theorem 14. *Suppose H is a graph and W is graphon such that $t(H,W) = 0$. Then W is countably partite.*

In Theorem 22 we shall see another "avoidance" result, this time for limits of permutations.

Lets turn the the last paper in Part II which concerns fractional isomorphism. For finite graphs the concept of fractional isomorphism was introduced by Tinhofer in 1986 [32] as a relaxation of graph isomorphism, with many additional favorable features such as a linear programming formulation and low complexity from the computational perspective. Remarkably, there are many different looking but equivalent definitions of fractional isomorphism. Grebík and Rocha [13] introduced the concept of fractional isomorphism for graphons and proved an analogous equivalence of counterparts of all these different characterizations. It follows from one of these characterizations that if $(G_n)_n$ and $(H_n)_n$ are two sequences, such that each G_n is fractionally isomorphic to H_n, $(G_n)_n$ converges in the cut distance to a graphon U and $(H_n)_n$ converges in the cut distance to a graphon W then U and W are fractionally isomorphic. Answering the main open question from [13], the main result of [P9] is the converse of this result.

Theorem 15. *Suppose that U and W are fractionally isomorphic graphons. The there exist sequences $(G_n)_n$ and $(H_n)_n$ of graphs such that each G_n is fractionally isomorphic to H_n, $(G_n)_n$ converges in the cut distance to a graphon U and $(H_n)_n$ converges in the cut distance to a graphon W.*

Part III: *Inhomogeneous random structures from graphons*

Theorem 1 provides a tool of studying "any sequence of graphs" using the theory of graphons. But we can take a dual approach and create random graph models from graphons, thus studying "typical sequences of graphs".

The most basic such model is the inhomogeneous random graph $\mathbb{G}(n, W)$ for $n \in \mathbb{N}$ and $W \in \mathcal{W}_0$.[3] This model, introduced in [25] is defined as follows. The vertex set is $V = \{1, \ldots, n\}$. We sample independently points $x_1, \ldots, x_n \in [0, 1]$ at random from λ. Then, independently for each $ij \in \binom{V}{2}$, we insert edge ij in $\mathbb{G}(n, W)$ with

[3]Let us note that the term *inhomogeneous random graphs* is overused and thus ambiguous. In other contexts, this term may refer for example to "exponential random graphs" or "geometric random graphs".

probability $W(x_i, x_j)$. Note that if W is constant-p, we get the binomial Erdős–Rényi random graph $\mathbb{G}(n,p)$, arguably the most studied random discrete structure. Recall also that most problems regarding $\mathbb{G}(n,p)$ are interesting and difficult when p is not a constant but rather a function $p = p(n)$ going to 0 as $n \to \infty$. This includes the problems of containment of a fixed graph, connectivity, hamiltonicity or the giant component, to name a few. This feature of vanishing probabilities is not available (at least in the basic version) in $\mathbb{G}(n,W)$, as the graphon W does not change with n. What I find fascinating, however, is that some graphons W can contain sparser and sparser bits and problems concerning $\mathbb{G}(n,W)$ have to deal with them with a similar level of difficulty as in sparse Erdős–Rényi random graphs.

Paper [P10] establishes a limit theorem for the count of fixed-sized cliques in $\mathbb{G}(n,W)$. To compare, we recall results of Bollobás [5], Ruciński [30], and Nowiscki and Wierman [28] from the 1980s that in particular establish a central limit theorem for the number of copies of a fixed-sized clique K_r in $\mathbb{G}(n,p)$, where $p > 0$ is fixed. Here, the variance is of the order n^{r-1}. In [P10] we establish a similar result for $\mathbb{G}(n,W)$. It turns out that there are actually four regimes, depending on the graphon W. Let us write X_n for the number of K_rs in $\mathbb{G}(n,W)$.

- If $W \equiv 0$ or $W \equiv 1$, then X_n does not have any variance,
- (X_n) satisfies the central limit theorem with variance of the order $n^{r-0.5}$,
- (X_n) satisfies the central limit theorem with variance of the order n^{r-1},
- $\left(\frac{X_n - \mathbb{E}[X_n]}{n^{r-1}}\right)$ converges to a certain chi-square distribution.

Let us note that since the publication of [P10], it has been significantly extended in [3].

Whereas [P10] deals with fixed-sized cliques in $\mathbb{G}(n,W)$, paper [P11] deals with the size of the largest clique $\mathbb{G}(n,W)$, denoted $\omega\left(\mathbb{G}(n,W)\right)$. Early (and easy) results of Matula [27] and Grimmett and McDiarmid [15] show that for p constant, we have

$$\omega\left(\mathbb{G}(n,p)\right) = (1 + o(1)) \cdot \frac{2\log_2 n}{\log_2(1/p)} .$$

Can we get a similarly compact description of $\omega(\mathbb{G}(n,W))$ for every graphon W? The next result from [P11] shows that we cannot.

Theorem 16. *Let $f:\mathbb{N}\to\mathbb{R}$ be an function with $\lim_{n\to\infty} f(n)=\infty$. Then there exists a graphon W and a sequence of integers $\ell_1 < k_1 < \ell_2 < k_2 < \ell_3 < k_3 < \ldots$ such that asymptotically almost surely,*

$$\omega(\mathbb{G}(\ell_i,W)) < f(\ell_i)\ , \textit{and}$$
$$\omega(\mathbb{G}(k_i,W)) > \frac{k_i}{f(k_i)}\ .$$

In plain words, the clique number of $\mathbb{G}(n,W)$ grows arbitrarily slowly and arbitrarily quickly at the same time! However, we can still obtain a pretty general result, which covers all graphons with a strictly positive essential infimum. That is, for each such graphon W, we identify a constant $\kappa(W)\in(0,+\infty]$ by

$$\kappa(W)=\sup\left\{-\frac{2\,\|h\|_1^2}{\int_{(x,y)\in[0,1]^2} h(x)h(y)\log_2 W(x,y)}\right\}\ ,$$

where the supremum ranges over all nonnegative functions $h\in L^1([0,1])$. The main theorem of [P11] is then as follows.

Theorem 17. *Suppose that W is a graphon with a strictly positive essential infimum.*

- *If $\kappa(W)<+\infty$ then asymptotically almost surely, $\omega(\mathbb{G}(n,W)) = (1+o(1))\cdot\kappa(W)\cdot\log_2 n$, and*
- *if $\kappa(W)=+\infty$ then asymptotically almost surely, $\omega(\mathbb{G}(n,W)) \gg \log_2 n$.*

We now turn to [P12] which concerns the uniform spanning tree. Recall that given a connected graph G the *uniform spanning tree in G* is just the uniform distribution over the (nonempty) set of all spanning spanning trees. The uniform spanning tree (UST) holds significant importance in mathematics due to its application in various fields such as probability theory, graph theory, and statistical physics. In probability theory, USTs are fundamental objects for studying random walks and percolation processes. In graph theory, they provide insights into the structure and connectivity of networks. Moreover, in statistical physics, USTs serve as models for understanding phase transitions and critical phenomena. Most research on the UST has been done on bounded-degree lattices such as the d-dimensional grid. This line of research includes the celebrated work on the Schramm–Loewner evolution. In [P12] we look at the UST in dense graphs. This area was unexplored prior, with two notable exceptions regarding the special case $G = K_n$. Firstly, Kolchin [21] and independently Grimmett [14] proved that the local structure of the UST of K_n (as $n \to \infty$) converges locally the the Galton–Watson branching process with offspring distribution $Poisson(1)$ conditioned on survival. Here, the notion of "local convergence" is often also called the "Benjamini–Schramm" convergence. Secondly, Aldous [1, 2] proved convergence in a global sense, leading to the notion of a "scaling limit" and "the continuum random tree". The main result of [P12] generalizes the former result from sequence $(K_n)_n$ to arbitrary sequence $(G_n)_n$ which converge to a given graphon W.

Theorem 18. *Suppose that W is a graphon with a positive minimum degree. Suppose that $(G_n)_n$ is a sequence of connected graphs converging to W. Then asymptotically almost surely, the USTs of G_n converge locally to a certain branching process derived from W and denoted τ_W.*

In [P12] we provide an explicit description of τ_W and based on this we can also derive some extremal results about the degree distribution of the UST in dense graphs.

Theorem 19. *For every $k \in \mathbb{N}$ and for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a connected graph on $n \geq n_0$ vertices with minimum degree at least ϵn. Write L_k for the random variable counting the number of vertices of degree k in a UST of G. Then*

- *(if $k = 1$)* $\mathbb{P}\left[L_1 \leq \left(e^{-1} - \epsilon\right) n\right] < \epsilon$,
- *(if $k = 2$)* $\mathbb{P}\left[L_2 \geq \left(e^{-1} + \epsilon\right) n\right] < \epsilon$,
- *(if $k \geq 3$)* $\mathbb{P}\left[L_k \geq \left(\frac{(k-2)^{k-2}}{(k-1)!e^{k-1}} + \epsilon\right) n\right] < \epsilon$.

All these bounds are best possible.

The last two papers in Part III initiate a theory of what we call "flip processes". We first cover the foundations given in [P13] and return applications of these foundations to specific flip processes in [P14] later. Suppose that $k \in \mathbb{N}$. We write $\mathcal{H}_k$ for the set of all $2^{\binom{k}{2}}$ graphs on the vertex set $\{1, \dots, k\}$. Let $\mathcal{R} = (\mathcal{D}_H)_{H \in \mathcal{H}_k}$ be a collection of probability distributions on $\mathcal{H}_k$, one distribution for each graph in $\mathcal{H}_k$. Suppose that G_0 is a given graph of order $n \geq k$. The *flip process with rule $\mathcal{R}$ and initial graph G_0* is a discrete time stochastic Markov chain $G_0, G_1, G_2, \dots$. For each $n \in \mathbb{N}$ the graph G_n is defined from $G_{n-1}$by taking a uniformly random k-tuple $(v_1, \dots, v_k) \subset V(G_{n-1})$ and replacing the graph $H := G_{n-1}[v_1, \dots, v_k]$ by a graph sampled from $\mathcal{D}_H$ while keeping all the other edges and nonedges as they were in G_{n-1}.

Flip processes are hence a broad class of random graph processes, essentially containing the Erdős–Rényi random graph process and the triangle removal process. Let us recall these two processes. In the Erdős–Rényi random graph process, one starts with the edgeless graph on n vertices, and in each steps turns a uniformly chosen nonedge into an edge. The process stops after $\binom{n}{2}$ many steps at which point the graph is complete. In the triangle removal process, one starts with the complete graph on n vertices, and in each steps removes the three edges of a uniformly selected triangle. The process stops at a random time in which the graph is triangle-free. The Erdős–Rényi random graph process is a common concept, a dynamical version of the Erdős–Rényi random graph. The triangle removal process was introduced by Bollobás and Erdős around 1990, and motivated in part by the offdiagonal Ramsey number $R(3, \ell)$. Of particular attention was the question of the stopping time, or equivalently, the number of edges in the final graph. Work on this question culminated in [4] where it is shown that the final graph has $n^{3/2+o(1)}$ edges with high probability.

Now consider $k_{ER} = 2$ and both for $H = edge$ and $H = nonedge$ let $\mathcal{D}_H$ be the Dirac distribution on the edge. If we consider the corresponding flip process $\mathcal{R}_{ER}$ started from an edgeless graph, we essentially get the Erdős–Rényi random graph process. The only difference is that the process $\mathcal{R}_{ER}$ is slowed down in adding individual edges at random times; this slowdown is almost nonexistent initially but intensifies as the graph gets denser. Next, consider $k_{TR} = 3$. For $H = K_3$, let $\mathcal{D}_H$ be the Dirac distribution on the edgeless graph. For $H \neq K_3$, let $\mathcal{D}_H$ be the Dirac distribution on H. The corresponding flip process $\mathcal{R}_{TR}$ started from a complete graph is a slowed down version of the triangle removal process.

We want to study the evolution of an initial n-vertex graph in a flip process with a fixed rule $\mathcal{R}$ of order k, as $n \to \infty$ with the lenses of dense graph limits. It is important to realize that as dense graph limits are sensitive only to changes of $\Theta(n^2)$ edges and we edit at most $\binom{k}{2} = O(1)$ edges in a single step, we will notice an change only in $\Theta(n^2)$ steps. The main result of [P13], which we present in a simplified and somewhat informal way, indeed says that there is a reasonable graphon desription of a typical evolution of a flip process started from any large graph.

Theorem 20. *Suppose that $\mathcal{R}$ is a rule of a flip process. Then there exist "trajectories" $\Phi : \mathcal{W}_0 \times [0, +\infty) \to \mathcal{W}_0$ with the following property. Suppose that $T > 0$. Suppose that G_0 is an n-vertex graph and U is a graphon such that the graphon representation W_0 of G_0 satisfies $d_\square(W_0, U) = o(1)$.*

Consider the flip process $(G_n)_n$ started from G_0. Write W_n for the graphon representation of G_n. Then with high probability, we have

$$\max\left\{d_\square\left(W_\ell, \Phi(U, \ell/n^2)\right) : \ell \in \mathbb{N} \cap \left[0, Tn^2\right]\right\} = o(1) .$$

That is, the theorem relates typical evolution of a discrete stochastic process to a deterministic analytic trajectory Φ. Let us describe how Φ is constructed, which is also the key for proving the theorem. Based on the rule $\mathcal{R}$, we construct a "velocity operator" $\mathfrak{V} : \mathcal{W} \to \mathcal{W}$ and define $\Phi(U, \cdot)$ as the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(U, t) = \mathfrak{V}\left(\Phi(U, t)\right) , \tag{4}$$

with the initial condition $\Phi(U, 0) = U$. The idea of describing a random discrete evolution using differential equations is not entirely new. In particular, Wormald [33] developed a framework of the "differential equations method" which has been applied to analyze dozens of randomized algorithms and random discrete structures. However, the main difference is that Wormald's machinery works with real-valued differential equations, whereas (4) is a differential equation in the Banach space $\mathcal{W}$.

In [P13] we establish many properties of trajectories. My favorite result is a construction (or rather a somewhat nonconstructive argument) of a periodic trajectory using the Poincaré–Bendixson theorem.

In [P14] we study seven specific families of flip processes, namely "ignorant", "balanced stirring", "complementing", "extremist", "monotone", "component completion", and "removal" flip processes. For example, in the extremist flip process of order k, the sampled graph $H := G_{n-1}[v_1, \ldots, v_k]$ is replaced by its complement (that is, edges are switched to nonedges and *vice versa*). In the extremist flip process, we replace H by the complete graph K_k if $e(H) \geq \frac{1}{2}\binom{k}{2}$ and by the edgeless graph I_k otherwise. The last mentioned family of removal flip processes, which generalizes from triangle removal to general F-removal, is the most interesting one. The sample theorem from [P14] given below is also the most interesting one mathematically, combining the compactness theorem and the graph removal lemma in its proof. In combination with Theorem 20 it suggests that if F_0 and G_0 are two graphs which are close in the cut distance and we run the triangle removal flip process either from F_0 or from G_0 then the resulting triangle-free graphs will be with high probability close in the cut distance.

Theorem 21. *Consider the triangle removal flip process. Let* $\Phi : \mathcal{W}_0 \times [0, +\infty) \to \mathcal{W}_0$ *be its trajectories. For each* $U \in \mathcal{W}_0$*, define* $destination(U) = \lim_{t\to\infty} \Phi(U, t)$*. Then the map* $U \mapsto destination(U)$ *is continuous with respect to the cut distance.*

In Part IV: *Limits of other discrete structures*

Following the success of the theory of dense graph limits (and the theory of sparse graph limits which has been developing in parallel), numerous other limit theories of discrete structures were worked out. A general theory of "flag algebras" developed by [29] let to solutions of dozens of problems in extremal combinatorics. However, the limit objects of flag algebras are abstract and somewhat elusive lacking the explicitness of graphons.[4] One of the most impressive limit theories was developed by Elek and Szegedy [10] for hypergraphs of a fixed uniformity. The existence of the limit object was shown by means of nonstandard analysis, with a combinatorial approach appearing only later [34]. Another successful limit theory was that of permutations. Here, by a *permutation* we mean a bijection $\pi : [n] \to [n]$. The difference to a notion of permutations on an abstract n-element set is that the elements of $[n]$ are equipped with the standard linear order. This linear order is instrumental in defining homomorphism densities, which like in the graph case, define the convergence. That is, for a permutation $A : [k] \to [k]$

[4] Note that an attempt to make flag albebras more explicit was recently taken by Coregliano and Razborov [9].

(with $k \leq n$) we define the *homomorphism density from A to π* by

$$t(A,\pi) = \frac{\#\text{sets } X \in \binom{[n]}{k} \text{ such that } \pi_{\restriction X} \text{ is compatible with } A}{\binom{n}{k}} \ ,$$

where the function $\pi_{\restriction X} : X \to [n]$ is *compatible with A* if the relative orders of the images in $\pi_{\restriction X}$ and A are the same. The theory of limits of permutations was worked out by Hoppen, Kohayakawa, Moreira, Ráth, and Sampaio [18]. The limit objects, called *permutons*, are Borel probability measures on $[0,1]^2$ whose marginals on both coordinates are the respective 1-dimensional Lebesgue measures. Homomorphism density extends to permutons as follows. Suppose that Γ is a permuton and A is as above. Sample k points $(x_1,y_1),\ldots,(x_k,y_k)$ from Γ independently at random. Let π be the random permutation defined so that $\pi(i)$ is the relative position on the y-coordinate of y_h (among $y_1,\ldots,y_k$), where h is selected so that the relative position on the x-coordinate of x_h is i. Then define $t(A,\Gamma) = \mathbb{P}[A = \pi]$.

Paper [P17] deals with permutons having zero density of a fixed pattern, also called *pattern avoiding*. This is a limit counterpart of a long line of research of pattern avoidance in permutations. The most important fruit of that research is a theorem of Marcus and Tardos [26], previously known as the Stanley–Wilf conjecture, which asserts that for any pattern A, the number of A-avoiding permutation on n elements is $\exp(O(n))$, compared to the factorial growth of all permutations. The main result of [P17] is that permutons avoiding any pattern must have 1-dimensional structure.

Theorem 22. *Suppose that A is a permutation of order k and Γ is a permuton with $t(A,\Gamma) = 0$. Consider the disintegration $(\Gamma_x)_{x\in[0,1]}$ of the measure Γ on the x-coordinate. Then almost every Γ_x is a convex combination of at most $k-1$ Dirac measures.*

This result is a permuton counterpart to a result of Cooper [8]. Also, note that there is some similarity to Theorem 14, although the proofs are very different. Using Theorem 22, we reprove a permutation removal lemma of Klimošová and Kráľ [20] which was also reproved by Fox and Wei [11].

Theorem 23. *For every pattern A and every $\epsilon > 0$ there exists $\delta > 0$ so that the following holds. Suppose that π is a permutation of order n with $t(A,\pi) < \delta$. Then there exists a permutation τ of order n which is avoiding A and for which we have $\sum_{i=1}^{n} |\pi(i) - \tau(i)| < \delta n^2$.*

While our proof does not give any effective bounds on δ in terms of A and ϵ, it is computationally much simpler than that those of Klimošová and Kráľ and Fox and Wei..

Next, we turn to limits of partial sets (posets) and recall a theory introduced by Janson [19]. Suppose that P is a set of size n and $<_P$ is a partial order on P. We define the *homomorphism density* $t((Q,<_Q),(P,<_P))$ from another poset $(Q,<_Q)$ (of size $k \leq n$) to $(P,<_P)$ as the probability that a random injective map from Q to P is a poset homomorphism. As for the space of limit objects, Janson considered ordered probability spaces $(\mathcal{S},\mathcal{F},\mu,\prec)$. That is, $(\mathcal{S},\mathcal{F},\mu)$ is a probability space, $\prec$ is a partial order on $\mathcal{S}$ so that $\{(x,y)\in\mathcal{S}\times\mathcal{S} : x \prec y\}$ is measurable with respect to $\mathcal{F}\times\mathcal{F}$. A *poset kernel* $W:\mathcal{S}\times\mathcal{S}\to[0,1]$ is a function such that for $x,y,z\in\mathcal{S}$, we have

- if $W(x,y)>0$ then $x \prec y$, and
- if $W(x,y)>0$ and $W(y,z)>0$ then $W(x,z)=1$.

Janson also extended the definition of homomorphism density from a finite poset to a poset kernel in a straightforward manner. One of the main results of [19] is a compactness theorem.

Theorem 24. *Suppose that* $((P_n,<_{P_n}))_n$ *is a sequence of posets of growing orders. Then there exists a subsequence* $\left((P_{n_i},<_{P_{n_i}})\right)_i$*, an ordered probability space* $(\mathcal{S},\mathcal{F},\mu,\prec)$ *and a poset kernel* $W:\mathcal{S}\times\mathcal{S}\to[0,1]$ *so that* $\left((P_{n_i},<_{P_{n_i}})\right)_i$ *converges to* W*.*

Answering an open question from [19] the main result of [P16] is that for $(\mathcal{S},\mathcal{F},\mu,\prec)$ we can always take the unit interval equipped with the Lebesgue measure and the standard linear order on $\mathbb{R}$. Working in this new setting actually leads to the limit being uniquely defined.

We now turn to the last paper in Part IV. Whereas [P16] and [P17] were contributions to existing limit theories, in [P15] we develop a new theory of Latin squares. A *Latin square* is an $n\times n$ matrix so that the entries of every row and every column give a permutation. Like in the setting of permutations, we work in the ordered setting, that is the columns of the matrix go from left to right, the rows go from up and bottom, and certain entries are bigger or smaller than others. The theory we develop has all features one expects from a limit theory.

- We construct limit objects, called “Latinons”.

- We define density $t(A, \cdot)$ of a "pattern" A into finite Latin squares and into Latinons. Convergence of a sequence $(L_n)_n$ of Latin squares to a Latinon K is defined by $\lim_n t(A, L_n) = t(A, K)$ for every pattern A.

- We prove the compactness theorem, that is, for every sequence of $(L_n)_n$ of Latin squares of growing orders there exists a subsequence $(L_{n_i})_i$ and a Latinon which is its limit.

- We introduce cut distance $\delta_\boxtimes$ for Latinons and prove that it is topologically equivalent to the convergence defined by densities.

- We prove that for every Latinon there exists a sequence of Latin squares converging to it.

We only detail some items from the above list, referring the reader to the full text of [P15] for the more technical ones. Let us explain the notion of densities into Latin squares, which defines the convergence. A *pattern* is a $k \times \ell$ matrix whose entries contain numbers $\{1, \ldots, k\ell\}$, each number exactly once. Given a pattern A of size $k \times \ell$ and a Latin square L of order $n \geq \max(k, \ell)$ we define density $t(A, L)$ as follows. Select a uniformly random set R of k rows of L. Select a uniformly random set C of ℓ columns of L. Then $t(A, L)$ as the probability that the relative values of $L_{\restriction R \times C}$ are as in A.

We do not give the definition of Latinons here as it is rather convoluted but we will try to hint at least one key element by making a comparison to graphons. A graphon $W : [0, 1]^2 \to [0, 1]$ arise as a limit of adjacency matrices of finite graphs. For each $(x, y) \in [0, 1]^2$, the value $W(x, y)$ corresponds to the proportion of 1s in the adjacency matrix (say, of order n) around the xn-th row and the yn-th column. With this in mind a Latinon at position $(x, y) \in [0, 1]^2$ should record what goes on in a finite Latin square of order n around the xn-th row and the yn-th column. This part of the Latin is now much more complex than in the case of adjacency matrices; rather than just containing 0s and 1s, it may contain all integers $\{1, \ldots, n\}$, which we think of after rescaling by n as numbers in the interval $[0, 1]$. In particular, the Latinon "at position (x, y)" should be a probability distribution on $[0, 1]$. The consideration so far hints to Latinons as functions $K : [0, 1]^2 \to \mathfrak{P}([0, 1])$ (where $\mathfrak{P}([0, 1])$ is the set of Borel probability measures on $[0, 1]$) satisfying some marginal conditions which are limit counterparts to the defining properties of Latin squares. This is a good starting point, but the actual limit object is more complicated than that.

References

[1] David Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[2] David Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.

[3] Bhaswar B. Bhattacharya, Anirban Chatterjee, and Svante Janson. Fluctuations of subgraph counts in graphon based random graphs. *Combin. Probab. Comput.*, 32(3):428–464, 2023.

[4] T. Bohman, A. Frieze, and E. Lubetzky. Random triangle removal. *Adv. Math.*, 280:379–438, 2015.

[5] Béla Bollobás. Threshold functions for small subgraphs. *Math. Proc. Cambridge Philos. Soc.*, 90(2):197–206, 1981.

[6] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.*, 219(6):1801–1851, 2008.

[7] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math. (2)*, 176(1):151–219, 2012.

[8] Joshua N. Cooper. A permutation regularity lemma. *Electron. J. Combin.*, 13(1):Research Paper 22, 20, 2006.

[9] L. N. Coregliano and A. A. Razborov. Semantic limits of dense combinatorial objects. *Uspekhi Mat. Nauk*, 75(4(454)):45–152, 2020.

[10] Gábor Elek and Balázs Szegedy. A measure-theoretic approach to the theory of dense hypergraphs. *Adv. Math.*, 231(3-4):1731–1772, 2012.

[11] J. Fox and F. Wei. Permutation property testing under different metrics with low query complexity. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1618–1637. SIAM, Philadelphia, PA, 2017.

[12] A. Frieze and R. Kannan. Quick Approximation to Matrices and Applications. *Combinatorica*, 19(2):175–220, 1999.

[13] J. Grebík and I. Rocha. Fractional isomorphism of graphons. *Combinatorica*, 42(3):365–404, 2022.

[14] G. R. Grimmett. Random labelled trees and their branching networks. *J. Austral. Math. Soc. Ser. A*, 30(2):229–237, 1980/81.

[15] G. R. Grimmett and C. J. H. McDiarmid. On colouring random graphs. *Math. Proc. Cambridge Philos. Soc.*, 77:313–324, 1975.

[16] H. Hatami. Graph norms and Sidorenko's conjecture. *Israel J. Math.*, 175:125–150, 2010.

[17] P.E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[18] C. Hoppen, Y. Kohayakawa, C. G. T. de A. Moreira, B. Ráth, and R. M. Sampaio. Limits of permutation sequences. *J. Comb. Theory, Ser. B*, 103(1):93–113, 2013.

[19] S. Janson. Poset limits and exchangeable random posets. *Combinatorica*, 31(5):529–563, 2011.

[20] T. Klimošová and D. Král'. Hereditary properties of permutations are strongly testable. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1164–1173. ACM, New York, 2014.

[21] V. F. Kolchin. Branching processes, random trees, and a generalized scheme of arrangements of particles. *Mathematical notes of the Academy of Sciences of the USSR*, 21(5):386–394, May 1977.

[22] J. Komlós. Tiling Turán theorems. *Combinatorica*, 20(2):203–218, 2000.

[23] D. Král', T. Martins, P. P. Pach, and M. Wrochna. The step Sidorenko property and non-norming edge-transitive graphs. *J. Combin. Theory Ser. A*, 162:34–54, 2019.

[24] L. Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.

[25] L. Lovász and B. Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.

[26] A. Marcus and G. Tardos. Excluded permutation matrices and the Stanley–Wilf conjecture. *J. Comb. Theory, Series A*, 107(1):153–160, 2004.

[27] D. W. Matula. The largest clique size in a random graph. Technical report, Department of Computer Science, Southern Methodist University, 1976.

[28] Krzysztof Nowicki and John C. Wierman. Subgraph counts in random graphs using incomplete U-statistics methods. In *Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986)*, volume 72, pages 299–310, 1988.

[29] A. A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[30] Andrzej Ruciński. When are small subgraphs of a random graph normally distributed? *Probab. Theory Related Fields*, 78(1):1–10, 1988.

[31] J. Skokan and L. Thoma. Bipartite subgraphs and quasi-randomness. *Graphs Combin.*, 20(2):255–262, 2004.

[32] G. Tinhofer. Graph isomorphism and theorems of Birkhoff type. *Computing*, 36(4):285–300, 1986.

[33] Nicholas C Wormald. Differential equations for random processes and random graphs. *Ann. Appl. Probab.*, pages 1217–1235, 1995.

[34] Yufei Zhao. Hypergraph limits: A regularity approach. *Random Structures & Algorithms*, 47(2):205–226, 2015.